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THE DYNAMIC PLASTIC BEHAVIOR OF SIMPLY SUPPORTED SPHERICAL SHELLS

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Abstract—A theoretical study has been undertaken into the dynamic plastic response of simply supported shallow and deep spherical shells subjected to exponentially decaying pressure pulses. The shells were assumed to be made from a rigid perfectly plastic material and exact solutions were obtained for three approximate yield conditions.

NOTATION

$e_\theta, e_\phi, k_\theta, k_\phi$	membrane strains and curvature changes
h	$H/2R$
$m_\theta, m_\phi, n_\theta, n_\phi$	M_θ/M_0, M_ϕ/M_0, N_θ/N_0, and N_ϕ/N_0, respectively
p, p_0, p_s	RP/N_0, RP_0/N_0 and RP_s/N_0, respectively
$\bar{p}$	$p_0 - p_s$
r, z	coordinates defined in Fig. 1(a)
s	S/N_0
u_n, u_ϕ	normal and meridianal displacements defined in Fig. 1(a)
v, w	u_ϕ/R and u_n/R, respectively
v_f, w_f	final values of v and w, respectively
x, y	r/R and z/R, respectively
$2H$	shell thickness
M_0, N_0	$\sigma_0 H^2$ and $2\sigma_0 H$, respectively
$M_\theta, M_\phi, N_\theta, N_\phi$	Bending moments and membrane forces defined in Fig. 1(b)
P	external pressure defined in Fig. 1(b)
P_0	maximum value of an exponentially decaying pressure pulse
P_s	static collapse pressure
R	base radius of a shallow shell as shown in Fig. 1(a) or radius of a deep spherical shell
S	transverse shear force defined in Fig. 1(b)
Z	total depth of a shallow shell
$\bar{Z}$	$2Z/R$
2α	subtended angle of a deep spherical shell
γ	$\dfrac{\rho R^2}{N_0 T_0^2}$
θ, ϕ	circumferential and meridianal coordinates of a spherical shell
$\kappa_\theta, \kappa_\phi$	$M_0 k_\theta/N_0$ and $M_0 k_\phi/N_0$, respectively
μ	$Z/(2H)$
ξ	$2Zx/R$
ρ	surface density of shell material
σ_0	tensile yield stress
τ	t/T_0, where t and T_0 are actual time and reference time, respectively
τ_f	t_f/T_0, where t_f is duration of motion
$(\dot{\ })$	$\dfrac{\partial}{\partial \tau}(\ )$
$(\ )'$	$\dfrac{\partial}{\partial x}(\ )$
$(\ )^*$	$\dfrac{\partial}{\partial \phi}(\ )$

INTRODUCTION

A STUDY was made in reference [1] into the behavior of a rigid perfectly plastic shallow shell of degree $n \geq 2$ which was loaded with a uniformly distributed static pressure. Exact theoretical solutions, according to the limit theorems of plasticity, were obtained for the three approximate yield conditions shown in Fig. 2, when the outer boundary of a shell of second degree was either simply supported or fully clamped.

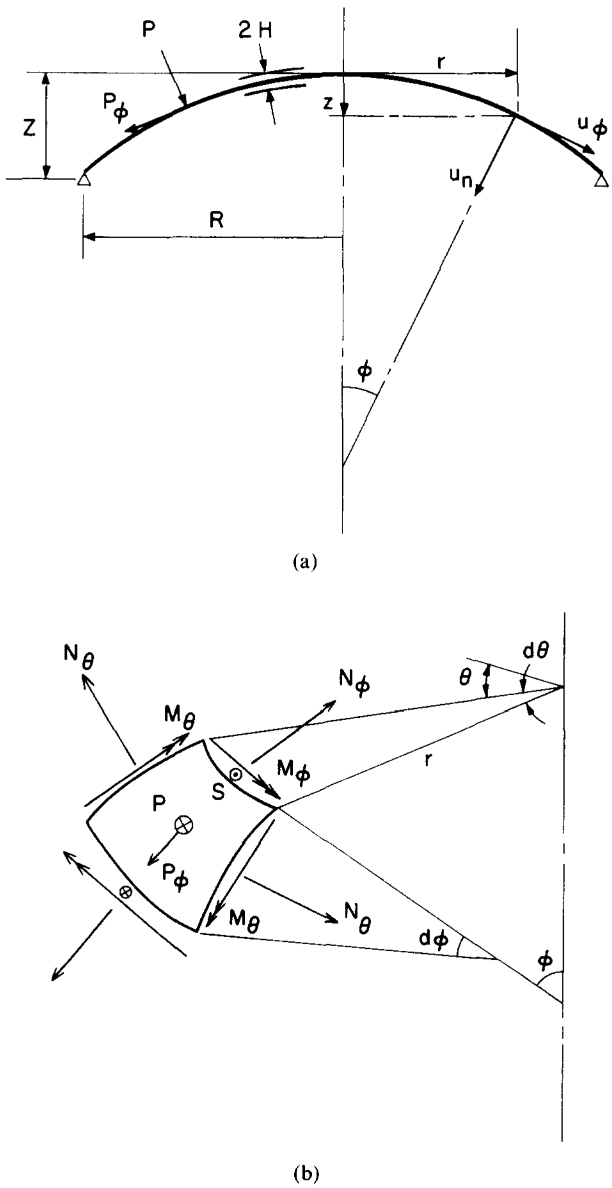


FIG. 1. (a) Shallow shell. (b) Stress resultants for a shallow shell.

In the first part of this article an investigation is conducted into the response of a simply supported rigid perfectly plastic shallow shell which is subjected to a uniformly distributed pressure $p = p_0 e^{-\tau}$. The peak value of the dynamic pressure (p_0) is larger than the corresponding static collapse pressure (p_s) which was obtained in reference [1]. Theoretical solutions are presented for shallow shells which are made from materials which obey the two moment limited interaction, uncoupled square and uncoupled diamond yield criteria shown in Fig. 2.

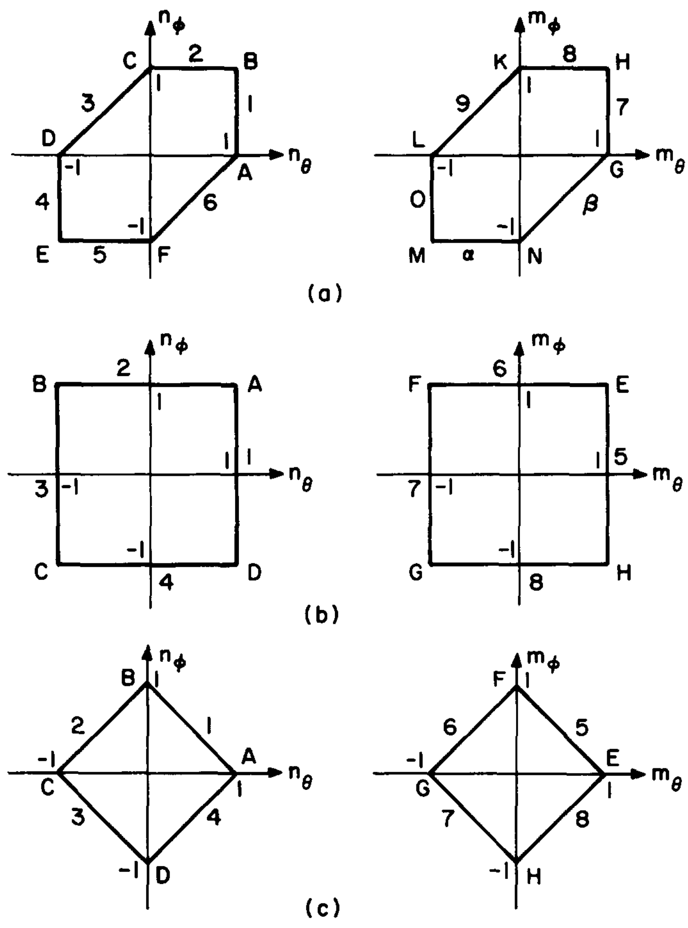


FIG. 2. (a) Two moment limited interaction yield conditions. (b) Uncoupled square yield condition. (c) Uncoupled diamond yield condition.

The dynamic plastic behavior of deep spherical shells is studied in the second part of this article. These shells are simply supported around the boundaries and are acted on by a uniformly distributed pressure $p = p_0 e^{-\tau}$.

BASIC RELATIONS

It was shown in Ref. [1] for rotationally symmetric shallow shells which undergo infinitesimal displacements that

$$x\dot{e}_\theta = \dot{v} - y'\dot{w}, \qquad \dot{e}_\phi = \dot{v}' - y''\dot{w} \tag{1a, b}$$

$$x\dot{\kappa}_\theta = -h(\dot{w}' + y''\dot{v}), \qquad \dot{\kappa}_\phi = (x\dot{\kappa}_\theta)' \tag{2a, b}$$

and

$$y''xn_\phi + y'n_\theta + (xs)' + xp = x\gamma\ddot{w} \tag{3a}$$

$$(xn_\phi)' - n_\theta - y''xs = x\gamma\ddot{v} \tag{3b}$$

$$xs = h[(xm_\phi)' - m_\theta] \tag{3c}$$

when normal loading (p) only is considered and provided rotary inertia is neglected.

DYNAMIC BEHAVIOR OF A SIMPLY SUPPORTED SHALLOW SHELL

The response of a shallow shell with a surface of revolution of second degree, which may be a shallow spherical, paraboloidal, ellipsoidal or hyperboloidal cap, is now examined when it is subjected to a uniformly distributed dynamic pressure

$$p = p_0\, e^{-\tau} \tag{4}$$

where $p = RP/N_0$, $p_0 = RP_0/N_0$ and $\tau = t/T_0$. It was shown in reference [1] that $y = Zx^n/R$ for a shell of degree n, where Z is the total depth of a shell. Thus,

$$y = Zx^2/R \tag{5}$$

for a shell second degree.

The dynamic behavior of a shallow shell which is made from a material which obeys the two moment limited interaction yield condition shown in Fig. 2(a) is examined in Section A(a). It transpires that this particular analysis is valid for pressures which lie within the range $0 \leq \bar{p} \leq 1{\cdot}2h$. However, it is shown in Section A(b) that the range of validity of this solution can be extended for pressures up to $6h$ when two additional plastic zones are allowed to develop in the shell. A shell then contains three distinct regions each of which is governed by a different portion of the yield condition. The range of validity of this theoretical solution is further extended in Section A(c).

Theoretical solutions are then presented in Sections B and C for the uncoupled square and uncoupled diamond yield criteria which are illustrated in Figs. 2(b) and 2(c), respectively.

A. *Two moment limited interaction yield condition*

(a) *Low pressure range* $0 \leq \bar{p} \leq 1{\cdot}2h$. It was shown in Ref. [1] that the régime 5–7 of the yield surface illustrated in Fig. 2(a) can be used to give the exact static collapse pressure

$$p_s = 6h + \frac{4Z}{R}. \tag{6}$$

If the peak value (p_0) of the dynamic pressure is slightly larger than the corresponding static collapse pressure (p_s) then it is anticipated that the whole shell will again collapse in the regime 5–7. Thus, the normality requirements of plasticity demand $\dot{e}_\theta = \dot{\kappa}_\phi = 0$, or

$$\dot{v} = y'\dot{w} \quad \text{and} \quad \dot{w}' + y''\dot{v} = \text{constant}. \tag{7a, b}$$

Equations (7a, b) have the solution

$$\dot{w} = \dot{w}_0(1-x) \quad \text{and} \quad \dot{v} = 2Zx\dot{w}_0(1-x)/R \tag{8a, b}$$

when using the shallow shell approximation and the kinematical requirements at $x = 1$ and where $\dot{w}_0$ is a function of time.

The equilibrium equations (3a, c) for the yield régime 5–7 (i.e., $n_\phi = -1$, $m_\theta = 1$) can be written

$$(xs)' = x\gamma\ddot{w}_0(1-x) + 2Zx/R - 2Zxn_\theta/R - xp \tag{9a}$$

$$n_\theta = -1 - 2Zxs/R - 2Zx^2\gamma\ddot{w}_0(1-x)/R \tag{9b}$$

and

$$xs = h[(xm_\phi)' - 1] \tag{9c}$$

when using equations (5), (8a) and 8(b). If equation (9b) is substituted into equation (9a) and the result integrated then it may be shown for a shallow shell that

$$xs = -(p - 4Z/R)x^2/2 + \gamma\ddot{w}_0(x^2/2 - x^2/3) \tag{10a}$$

and

$$n_\theta = -1 + 2Z\{(p - 4Z/R)x^2/2 - \gamma\ddot{w}_0(3x^2/2 - 4x^3/3)\}/R \tag{10b}$$

which allow equation (9c) to be integrated

$$m_\phi = 1 - (p - 4Z/R)x^2/(6h) + \gamma\ddot{w}_0(x^2 - x^3/2)/(6h). \tag{10c}$$

Now for a shallow shell with a simply supported edge, $m_\phi = 0$ at $x = 1$, so that equation (10c) gives

$$\gamma\ddot{w}_0 = 2(p - p_s) \tag{11a}$$

or,

$$\gamma\dot{w}_0 = 2\{p_0(1 - e^{-\tau}) - p_s\tau\} \tag{11b}$$

when using (4) and the initial condition $\dot{w}_0 = 0$ at $\tau = 0$. The motion of a shell ceases when $\gamma\dot{w}_0 = 0$ and the corresponding duration of motion (τ_f) is

$$p_0(1 - e^{-\tau_f}) - p_s\tau_f = 0. \tag{12}$$

If equation (11b) is integrated, then using (12) and $w_0 = 0$ at $\tau = 0$ it may be shown that the permanent transverse displacement is

$$\gamma w_f = 2\{\bar{p}\tau_f - p_s\tau_f^2/2\}(1-x) \tag{13}$$

while the permanent tangential displacement (γv_f) is $2Zx\gamma w_f/R$. The total inelastic energy absorbed by a shallow shell throughout deformation is

$$E_a = 2\pi R^2 N_0 \int_0^{\tau_f} p \, d\tau \int_0^1 \dot{w}x \, dx \tag{14a}$$

or

$$E_a = 2\pi R^2 N_0 p_s \tau_f(\bar{p} - p_s\tau_f/2)/(3\gamma). \tag{14b}$$

It is necessary to show that the above solution is kinematically admissible (i.e., $\dot{e}_\phi \leq 0$, $\dot{\kappa}_\theta \geq 0$) and statically admissible (i.e., $0 \leq m_\phi \leq 1$, $-1 \leq n_\theta \leq 0$) in order to be exact according to the theorems of plasticity for the yield surface selected. The kinematical requirements are satisfied for a shallow shell while the static requirements demand [2]

$$\bar{p} \leq 1{\cdot}2h \quad \text{and} \quad 12h + p_0 - p_s(1+\tau_f) \geq 0. \tag{15a, b}$$

If a shallow shell is subjected to a rectangular pressure pulse i.e.,

$$p = p_0 \quad \text{for } 0 \leq \tau \leq 1 \quad \text{and} \quad p = 0 \quad \text{when } \tau \geq 1 \tag{16}$$

then a solution can be obtained with the same yield régime and velocity field which was used in the previous case. Thus, equation (11a) gives

$$\gamma\dot{w}_0 = 2\bar{p}\tau \quad \text{if } 0 \leq \tau \leq 1 \tag{17a}$$

and

$$\gamma\dot{w}_0 = 2(p_0 - p_s\tau) \quad \text{when } \tau \geq 1. \tag{17b}$$

Clearly $\dot{w}_0 = 0$ when $\tau_f = p_0/p_s$ which may be used to show that the permanent transverse displacements are

$$\gamma w_f = p_0(p_0/p_s - 1)(1-x) \tag{18}$$

while the permanent tangential displacements (γv_f) are $2Zx\gamma w_f/R$. This solution is kinematically admissible for $0 \leq \tau \leq \tau_f$ while the static requirements demand [2]

$$0 \leq \bar{p} \leq 1{\cdot}2h \quad \text{and} \quad p_s \leq 12h. \tag{19a, b}$$

The inequality (19b) can be rewritten with the aid of (6) in the form $4Z/R \leq 6h$ or

$$Z \leq 3(2H)/8. \tag{20}$$

This latter requirement for a rectangular pressure pulse is very severe since the height of a shell must be less than $\frac{3}{8}$ of it thickness.

(b) *Pressure range* $1{\cdot}2h \leq \bar{p} \leq 6h$. If the static admissibility requirements outlined above for exponential pressure pulse loading (4) are examined in detail then it is observed that $n_\theta'' < 0$ at $x = 0$ when $\bar{p} > 1{\cdot}2h$. This suggested that the response for higher pressures could be obtained when employing the yield régimes E–7 and 5–7 for the shell regions $0 \leq x \leq u$ and $u \leq x \leq 1$, respectively. However, although this procedure gave a statically admissible solution, it was found to be kinematically inadmissible for the region $0{\cdot}8u \leq x \leq u$. The shell was studied finally using the yield régimes E–7, 4–7 and 5–7 for the shell regions $0 \leq x \leq u_1$, $u_1 \leq x \leq u_2$ and $u_2 \leq x \leq 1$, respectively.

(i) $0 \leq x \leq u_1$. It can be shown when using equation (5) that equations (3a) and (3b) with $n_\theta = n_\phi = -1$ can be arranged to give

$$\gamma\ddot{v} = -2ZY/(Rx) \quad \text{and} \quad \gamma\ddot{w} = Y'/x + p - 4Z/R \tag{21a, b}$$

where $Y = xs$. The flow rule requires $\dot{\kappa}_\phi = 0$, or

$$\dot{w}' + y''\dot{v} = \dot{C}_0/\gamma \tag{22}$$

where $\dot{C}_0$ is independent of x but may be a function of τ. If equations (21a) and (21b) are substituted into the time derivative of (22) then

$$\xi^2\frac{d^2Y}{d\xi^2}-\xi\frac{dY}{d\xi}-\xi^2Y=\frac{R^3\ddot{C}_0\xi^3}{8Z^3} \tag{23}$$

where $\xi = 2Zx/R$ and $Y = dY/d\xi = 0$ at $\xi = 0$. Equation (23) has the solution [2, 3].

$$Y=\frac{\ddot{C}_0R^3}{8Z^3}\left\{-\xi+\xi I_1\int_0^{\xi}K_0\,d\eta+\xi K_1\int_0^{\xi}I_0\,d\eta\right\}+\ddot{C}_1\xi I_1+\ddot{C}_2\xi K_1 \tag{24}$$

where I_n and K_n are modified Bessel functions of the first and second kind of order n, respectively. Now $\ddot{C}_2 = 0$ since $Y = 0$ at $\xi = 0$. Thus, using equations (21a, 21b and 24) and the initial conditions $w = \dot{w} = v = \dot{v} = 0$ at $\tau = 0$ gives

$$\gamma\dot{v}=-\frac{\dot{C}_0R}{2Z}\left\{-1+I_1\int_0^{\xi}K_0(\eta)\,d\eta+K_1\int_0^{\xi}I_0(\eta)\,d\eta\right\}-\frac{4Z^2\dot{C}_1I_1}{R^2} \tag{25a}$$

and

$$\gamma\dot{w}=p_0(1-e^{-\tau})-\frac{4Z\tau}{R}+\frac{\dot{C}_0R}{2Z}\left(I_0\int_0^{\xi}K_0\,d\eta-K_0\int_0^{\xi}I_0\,d\eta\right)+\frac{4Z^2\dot{C}_1I_0}{R^2} \tag{25b}$$

with $\dot{C}_0 = \dot{C}_1 = 0$ at $\tau = 0$. Equation (3c) may be integrated with the aid of (24) and $m_\theta = 1$ to give

$$\begin{aligned}h\xi m_\phi = h\xi+\frac{\ddot{C}_0R^3}{8Z^3}\Bigg[&-\frac{\xi^2}{2}+\xi I_0\int_0^{\xi}K_0\,d\eta-\xi K_0\int_0^{\xi}I_0\,d\eta\\&+\int_0^{\xi}K_0(\eta)\left\{\int_0^{\eta}I_0(\eta_1)\,d\eta_1\right\}d\eta-\int_0^{\xi}I_0(\eta)\left\{\int_0^{\eta}K_0(\eta_1)\,d\eta_1\right\}d\eta\Bigg]+\ddot{C}_1\int_0^{\xi}\eta I_1\,d\eta. \end{aligned}\tag{26}$$

It can be shown [2] when using the shallow shell approximation ($\xi^2 \ll 1$) and the expansion of I_n and K_n given in Ref. [3] that equations (24, 25a, 25b and 26) become

$$xs = x^2(2Z^2\ddot{C}_1/R^2+\ddot{C}_0x/3) \tag{27a}$$

$$\gamma\dot{v} = -2Zx(2Z^2\dot{C}_1/R^2+\dot{C}_0x/3)/R \tag{27b}$$

$$\gamma\dot{w} = p_0(1-e^{-\tau})-4Z\tau/R+4Z^2\dot{C}_1/R^2+\dot{C}_0x \tag{27c}$$

and

$$hxm_\phi = hx+2Z^2\ddot{C}_1x^3/(3R^2)+\ddot{C}_0x^4/12 \tag{27d}$$

for a shallow shell with $y'^2 = (2Zx/R)^2 = \xi^2 \ll 1$.

(ii) $u_1 \le x \le u_2$. This part of a shell is in the yield régime 4–7 for which $n_\theta = -1$, $m_\theta = 1$, $\dot{e}_\phi = 0$ and $\dot{\kappa}_\phi = 0$. Thus,

$$\dot{v}'-y''\dot{w}=0 \quad\text{and}\quad \dot{w}'+y''\dot{v}=\dot{A}/\gamma \tag{28a, b}$$

where $\dot{A}$ is independent of x but may be a function of τ. Equations (28a and 28b) may be solved to give

$$\gamma\dot{v} = (\dot{A}+\bar{Z}\dot{B})/\bar{Z}+\bar{Z}x(\dot{C}-\bar{Z}\dot{B}x/2) \tag{29a}$$

and

$$\gamma\dot{w} = \dot{C} - \bar{Z}\dot{B}x \tag{29b}$$

when using the shallow shell approximation and where $\dot{B}$ and $\dot{C}$ are unknown functions of τ.

The equations of motion (3a–3c) with $n_\theta = -1$ and $m_\theta = 1$ can be solved with the aid of (5) to give

$$\begin{aligned} xn_\phi = {} & \bar{Z}^{-3}\ddot{A} + \ddot{D} + (\bar{Z}\ddot{E} + 1 - p\bar{Z}^{-1})x + (\ddot{B} - \bar{Z}^2\ddot{D})x^2/2 \\ & + \bar{Z}(3\ddot{C} - \bar{Z}^2\ddot{E})x^3/6 + \bar{Z}^2(\bar{Z}^2\ddot{D} - 6\ddot{B})x^4/24 \end{aligned} \tag{30a}$$

$$\begin{aligned} xs = {} & \ddot{E} - \bar{Z}^{-2}\ddot{Q} - (\bar{Z}^{-2}\ddot{A} + \bar{Z}\ddot{D})x + (\ddot{C} - \bar{Z}^2\ddot{E})x^2/2 \\ & \bar{Z}(\bar{Z}^2\ddot{D} - 3\ddot{B})x^3/6 \end{aligned} \tag{30b}$$

and

$$\begin{aligned} hxm_\phi = {} & \ddot{F} + \bar{Z}^{-1}\ddot{D} - \bar{Z}^{-3}\ddot{B} + (h + \ddot{E} - \bar{Z}^{-2}\ddot{Q})x - (\bar{Z}^{-2}\ddot{A} + \bar{Z})x^2/2 \\ & + (\ddot{C} - \bar{Z}^2\ddot{E})x^3/6 + \bar{Z}(\bar{Z}^2\ddot{D} - 3\ddot{B})x^4/24 \end{aligned} \tag{30c}$$

when employing the shallow shell approximation and where $\ddot{D}$, $\ddot{E}$ and $\ddot{F}$ are unknown functions of time and

$$\dot{Q} = p_0(1 - e^{-\tau}) - (p_s - 6h)\tau \tag{31a}$$

or

$$\ddot{Q} = p - p_s + 6h. \tag{31b}$$

(iii) $u_2 \leq x \leq 1$. The analysis for this part of a shell is similar to that for the low pressure range considered in Section A(a). Thus,

$$\gamma\dot{w} = \dot{K}(x-1),\ \gamma\dot{v} = \bar{Z}\gamma\dot{w}x \tag{32a, b}$$

$$n_\theta = -1 - \bar{Z}\{\ddot{G} - h - (3\ddot{K} + \ddot{Q})x^2/2 + 4\ddot{K}x^3/3\} \tag{32c}$$

$$xs = \ddot{G} - h - (\ddot{K} + \ddot{Q})x^2/2 + \ddot{K}x^3/3 \tag{32d}$$

$$hxm_\phi = \ddot{K}/12 + \ddot{Q}/6 - \ddot{G}(1-x) - (\ddot{K} + \ddot{Q})x^3/6 + \ddot{K}x^4/12 \tag{32e}$$

where $\ddot{G}$ and $\dot{K}$ are unknown functions of time.

Now equations (27a–d, 29a, 29b, 30a–c and 32a–e) contain the 10 unknowns $\dot{C}_0$, $\dot{C}_1$, $\dot{A}$, $\dot{B}$, $\dot{C}$, $\ddot{D}$, $\ddot{E}$, $\ddot{F}$, $\dot{K}$ and $\ddot{G}$ which together with the two unknowns u_1 and u_2 can be determined from the continuity of n_ϕ, m_ϕ, s, $\dot{v}$, $\dot{w}$ and $\dot{w}'$ at $x = u_1$ and $x = u_2$. It should be noted that $\dot{w}'$ cannot be discontinuous at $x = u_1$ and $x = u_2$ since $|m_\phi| < 1$. It is observed [2] at $\tau = \tau_1$, which is given by

$$p_0(1 - e^{-\tau_1}) - p_s\tau_1 - 1{\cdot}2h\tau_1 = 0 \tag{33}$$

that the collapse régime 5–7 of the zone $u_2 \leq x \leq 1$ has spread to the entire shell (i.e., $u_1 = u_2 = 0$). Thus, the above theoretical analysis is valid during a first stage of motion ($0 \leq \tau \leq \tau_1$) and it is therefore necessary to consider a second stage of motion ($\tau_1 \leq \tau \leq \tau_f$). However, the analysis of the second stage of motion ($\tau_1 \leq \tau \leq \tau_f$) is similar to the low pressure case A(a) except that the initial conditions now correspond to those at the end of the first stage of motion when $\tau = \tau_1$. Therefore, equations (8a, 8b and 11a) remain valid for $\tau_1 \leq \tau \leq \tau_f$ and may be solved to give

$$\gamma \dot{w} = 2\{p_0(1-e^{-\tau})-p_s\tau\}(1-x) \tag{34}$$

when matching the velocity $\dot{w}_0$ at $\tau = \tau_1$ with equation (27c) and using

$$\dot{C}_0 = -2\{p_0(1-e^{-\tau})-p_s\tau\} \quad \text{and} \quad \bar{Z}^2\dot{C}_1 = p_0(1-e^{-\tau})-p_s\tau-6h\tau$$

which were obtained from the continuity requirements at $x = u_1$ and $x = u_2$. The motion of a shell terminates when $\dot{w} = 0$, or

$$p_0(1-e^{-\tau_f})-p_s\tau_f = 0. \tag{35}$$

The permanent transverse displacements and the energy absorbed during deformation are again given by equations (13) and (14b).

It has been shown in Ref. [2] that the above solution is kinematically and statically admissible provided

$$\bar{p} \leq 6h \quad \text{and} \quad 12h+p_0\,e^{-\tau_f}-p_s \geq 0. \tag{36a, b}$$

Equation (36b) has been examined numerically and found to be satisfied for a range of pressures $\bar{p} \leq 6h$ and for shell heights varying from 1 to 5 times the shell thickness.

(c) *Pressure range* $6h \leq \bar{p} \leq \lambda h$. A shallow shell which is subjected to a dynamic pressure pulse in this range collapses into the four regions $0 \leq x \leq u_0$, $u_0 \leq x \leq u_1$, $u_1 \leq x \leq u_2$ and $u_2 \leq x \leq 1$ which are governed by the yield régimes E–H, E–7, 4–7 and 5–7 of Fig. 2(a), respectively.

It is straightforward to show in the region $0 \leq x \leq u_0$ that

$$\gamma \ddot{w} = \ddot{Q} \quad \text{or} \quad \gamma \dot{w} = \dot{Q} \tag{37a, b}$$

where $\dot{Q}$ is defined by equation (31b). The equations of equilibrium and flow rules for the remaining three regions are the same as those in the corresponding three zones in the first stage of motion of case A(b). Thus, the analysis of these regions is very similar to that outlined previously provided due account is taken of the boundary conditions at $x = u_0$. The various constants of integration which appear in these equations can be determined from the continuity requirements. This first stage of motion is completed when $u_0 = 0$ at $\tau = \tau_0$ where

$$p_0(1-e^{-\tau_0})-p_s\tau_0-6h\tau_0 = 0. \tag{38}$$

The second ($\tau_0 \leq \tau \leq \tau_1$) and third ($\tau_1 \leq \tau \leq \tau_f$) stages of motion are respectively similar to the first and second stages in case A(b). The maximum transverse displacement is

$$\gamma w_0 = 2\bar{p}\tau_f-p_s\tau_f^2+(p_s+6h)(1+\tau_0/2)\tau_0-p_0\tau_0 \tag{39}$$

where

$$p_0(1-e^{-\tau_f})-p_s\tau_f = 0. \tag{40}$$

It is shown in Ref. [2] that the solution outlined above is kinematically and statically admissible for a shallow shell provided $\bar{p}-p_s\tau_f+12h \geq 0$ or $\bar{p} \leq \lambda h$. The largest admissible peak value (p_0) of the exponentially decaying non-dimensionalized dynamic pressure is shown in Fig. 3 ($\lambda = p_{0H}-p_s$).

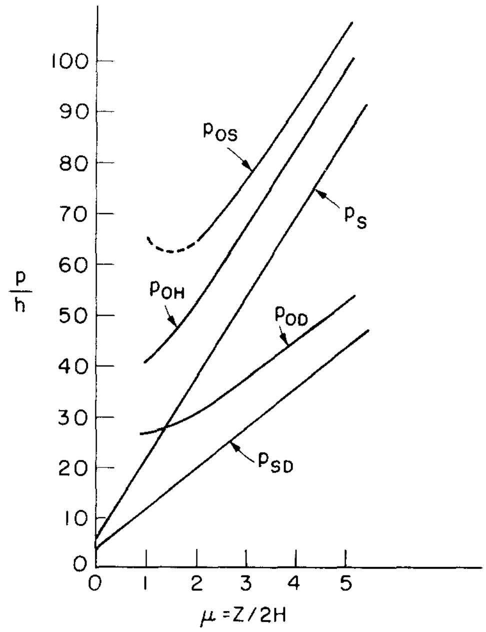


FIG. 3. Static collapse pressures of simply supported shallow shells and upper limits of the peak values of the dynamic pressures

p_s	static collapse pressures for yield surfaces in Fig. 2(a) and 2(b)
p_{sD}	static collapse pressures for yield surface in Fig. 2(c)
p_{OH}, p_{Os}, p_{OD}	upper limits of peak values of exponentially decaying dynamic pressures for yield surfaces in Figs. 2(a), 2(b) and 2(c), respectively.

It is evident that a theoretical analysis of a shallow shell which is subjected to a dynamic pressure $\bar{p} \geq \lambda h$ would involve at least 5 different yield régimes and four stages of motion. Thus, the dynamic behavior of a shallow shell is explored further when using the two simpler yield surfaces illustrated in Figs. 2(b) and 2(c).

B. *Uncoupled square yield condition*

It was shown in Ref. [1] that the collapse pressure of a simply supported shallow shell is the same for either of the two yield conditions illustrated in Figs. 2(a) and 2(b) (i.e., equation (6)). Moreover, when $0 \leq \bar{p} \leq 1{\cdot}2h$ then the whole shell lies in the yield régime 4–5 so that the behavior is again described by the same equations and results which are given in Section A(a). If $1{\cdot}2h \leq \bar{p} \leq 6h$, then the yield régimes C–5, 3–5 and 4–5 can be used for the zones $0 \leq x \leq u_1$, $u_1 \leq x \leq u_2$ and $u_2 \leq x \leq 1$, respectively. Thus, again the various expressions presented in Section A(b) remain valid for this case. When the initial value of a pressure pulse lies within the range $6h \leq \bar{p} \leq \bar{\lambda}h$ then the yield régimes C–E, C–5, 3–5 and 4–5 may be used in order to describe the behavior in the shell zones $0 \leq x \leq u_0$, $u_0 \leq x \leq u_1$, $u_1 \leq x \leq u_2$ and $u_2 \leq x \leq 1$, respectively. The theoretical analysis is identical to that outlined in Section A(c) except that the value of $\bar{\lambda}$ is different to λ since m_ϕ is now allowed to be negative. This gives a limit value p_{0L} of p_0 which is higher than that permitted by the two moment limited interaction yield criterion and may be determined from the equations

$$p_{0L}(1-e^{-\tau_f})-p_s\tau_f = 0 \tag{41a}$$

and

$$p_{0L}\,e^{-\tau_f}-p_s+18h = 0. \tag{41b}$$

C. *Uncoupled diamond yield condition*

It was shown in Ref. [1] that

$$p_s = 4h+\frac{2Z}{R} \tag{42}$$

is the exact static collapse pressure of a simply supported shallow shell made from a material which obeys the uncoupled diamond yield condition illustrated in Fig. 2(c). If the magnitude of a dynamic pressure pulse is not too large then the whole shell will deform in the yield régime 3–5 which is the same as that used to obtain equation (42). The flow rules and kinematic boundary conditions are the same as those in the corresponding static problem so that the velocity field is

$$\dot{v} = 0 \quad \text{and} \quad \dot{w} = \dot{w}_0(1-x^2) \tag{43a, b}$$

where $\dot{w}_0$ is a function of time.

The equilibrium equation (3a) when solved with the aid of $n_\theta+n_\phi = -1$ and (5) gives

$$xs = \gamma\ddot{w}_0(2x^2-x^4)/4-(p-\bar{Z})x^2/2. \tag{44}$$

If equation (44) and $m_\theta+m_\phi = 1$ are substituted into (3c) then

$$m_\phi = \tfrac{1}{2}+\gamma\ddot{w}_0(3x^2-x^4)/(24h)-(p-\bar{Z})x^2/(8h). \tag{45}$$

Now $m_\phi = 0$ at $x = 1$, or

$$\gamma\ddot{w}_0 = 3(p-p_s)/2 \tag{46}$$

which together with equations (5, 43a, 43b and 44) and the yield condition $n_\theta+n_\phi = -1$ allows (3b) to be integrated

$$n_\phi = -\tfrac{1}{2}+\bar{Z}x^2\{p-p_s-8h-(p-p_s)x^2\}/16. \tag{47}$$

It is straightforward to show that the generalized stresses can be written in the form

$$m_\phi = \tfrac{1}{2}+P(x),\; m_\theta = \tfrac{1}{2}-P(x) \tag{48a, b}$$

$$n_\phi = -\tfrac{1}{2}+\bar{Z}hP(x) \quad \text{and} \quad n_\theta = -\tfrac{1}{2}-\bar{Z}hP(x) \tag{48c, d}$$

where

$$P(x) = \{p-p_s-8h-(p-p_s)x^2\}x^2/(16h). \tag{49}$$

If equation (4) is substituted into (46) then

$$\gamma\dot{w}_0 = 1{\cdot}5\{p_0(1-e^{-\tau})-p_s\tau\} \tag{50}$$

since $\dot{w} = 0$ at $\tau = 0$. It is evident that motion ceases at $\tau = \tau_f$ where

$$p_0(1-e^{-\tau_f})-p_s\tau_f = 0. \tag{51}$$

The permanent displacement field from equations (43a, 43b, 50 and 51) is $\gamma v_f = 0$ and

$$\gamma w_f = 0{\cdot}75(2\bar{p}\tau_f-p_s\tau_f^2)(1-x^2) \tag{52}$$

while the total energy absorbed throughout deformation is

$$E_a = 3\pi R^2 N_0 p_s \tau_f(\bar{p} - p_s\tau_f/2)/(4\gamma). \tag{53}$$

It can be shown that the above theoretical analysis is kinematically admissible. In order for this solution to be also statically admissible it is necessary to ensure that the generalized stresses are restricted to the ranges $0 \le m_\phi \le 1$, $0 \le m_\theta \le 1$, $-1 \le n_\phi \le 0$ and $-1 \le n_\theta \le 0$. It is evident from equations (48a–d and 49) that these inequalities are satisfied provided $-1 \le 2P(x) \le 1$, or

$$p_0 e^{-\tau_f} - p_s + 8h \ge 0 \tag{54a}$$

for all values of $\bar{p}$ where τ_f is given by (51) and

$$\bar{p} \le 16(1{\cdot}5 + \sqrt{2})h \tag{54b}$$

when

$$\bar{p} \ge 8h.$$

DYNAMIC BEHAVIOR OF SIMPLY SUPPORTED DEEP SPHERICAL SHELLS

The behavior of a simply supported deep spherical shell which is subjected to a dynamic pressure field which is described by equation (4) is examined in this section.

It is straightforward to show [e.g. 2, 4] that

$$\dot{e}_\theta = \dot{v}\cot\phi - \dot{w}, \qquad \dot{e}_\phi = \dot{v}^* - \dot{w} \tag{55a, b}$$

$$\dot{\kappa}_\theta = -h\cot\phi(\dot{v} + \dot{w}^*) \tag{55c}$$

and

$$\dot{\kappa}_\phi = -h(\dot{v} + \dot{w}^*)^* \tag{55d}$$

while if a spherical shell is loaded only in the normal direction and rotary inertia is unimportant then

$$(n_\theta + n_\phi)\sin\phi + (s\sin\phi)^* + p\sin\phi = \gamma\ddot{w}\sin\phi \tag{56a}$$

$$(n_\phi \sin\phi)^* - n_\theta\cos\phi - s\sin\phi = \gamma\ddot{v}\sin\phi \tag{56b}$$

and

$$h\{(m_\phi\sin\phi)^* - m_\theta\cos\phi\} = s\sin\phi \tag{56c}$$

according to the principle of virtual work.

Theoretical solutions which are based on the two moment limited interaction and uncoupled square yield criteria are discussed in Sections A and B, respectively, while a solution using the uncoupled diamond yield condition is presented in Section C.

A. *Two moment limited interaction yield surface*

Onat and Prager [5] have presented upper and lower bounds to the exact static collapse pressure of a fully clamped spherical shell made from a material which obeys the Tresca yield criterion, while Hodge [6] has derived an exact solution for both the simply supported and fully clamped cases when using the two moment limited interaction yield surface. Sankaranarayanan [4, 7] also employed the two moment limited interaction yield condition in order to examine the dynamic plastic response of a simply supported spherical shell. These results have been discussed in Ref. [2] wherein a limited series expansion of Sankaranarayanan's solutions were undertaken in order to provide more insight into the range of validity. An attempt was made to extend the range of validity of this solution by including additional yield régimes but progress was impeded by the complexity of the algebra and the necessity of considering at least three yield régimes. Thus, further theoretical investigations into the dynamic behavior of deep spherical shells were pursued by employing the simpler yield conditions illustrated in Figs. 2(b) and 2(c).

B. *Uncoupled square yield surface*

It is straightforward to show when the equilibrium equations (56a–c) for static loading are solved with the appropriate relations for the portion 4–5 of the yield surface illustrated in Fig. 2(b) that

$$m_\phi = 1-\frac{(p-2)}{2h\sin\phi}\{\log(\sec\phi+\tan\phi)-\sin\phi\}. \tag{57}$$

Thus, the static collapse pressure of a simply supported shell ($m_\phi = 0$ at $\phi = \alpha$) is the same as that derived by Hodge [6], while

$$p_s = 2+4h\{\operatorname{cosec}\alpha\log(\sec\alpha+\tan\alpha)-1\}^{-1} \tag{58}$$

is the static collapse pressure of a fully clamped spherical shell with $m_\phi = -1$ at $\phi = \alpha$.

A theoretical analysis of a simply supported spherical shell which is subjected to small dynamic pressures is similar to the corresponding solution which was obtained by Sankaranarayan [4] who employed a two moment limited interaction yield condition. However, the maximum permitted value of $\bar{p}$ is now larger because m_ϕ can become -1 in the yield régime 4–5.

C. *Uncoupled diamond yield surface*

If a spherical shell has a simply supported edge, then an exact solution for the static collapse pressure can be obtained when using the yield régime 3–5 of Fig. 2(c), i.e.,

$$n_\theta+n_\phi = -1 \quad\text{and}\quad m_\theta+m_\phi = 1. \tag{59a, b}$$

The equilibrium equation (56a) can be integrated with the aid of (59a) to give

$$s\sin\phi = -(p-1)(1-\cos\phi) \tag{60}$$

while equations (56c) and (59b) yield

$$m_\phi = \frac{1}{2}-\frac{(p-1)}{2h}\frac{(1-\cos\phi)}{(1+\cos\phi)}. \tag{61}$$

Now $m_\phi = 0$ at $\phi = \alpha$, or from (61) the static collapse pressure is

$$p_s = 1+h\cot^2(\alpha/2) \tag{62}$$

which reduces to equation (42) for a shallow shell provided $\alpha^2 \ll 1$ and the non-dimensional variables are defined for a shallow shell. The solution of equation (56b) and equations (59a, 59b and 61) can be expressed in the form

$$n_\phi = -\tfrac{1}{2}-hF(\phi), \qquad n_\theta = -\tfrac{1}{2}+hF(\phi) \tag{63a, b}$$

$$m_\phi = \tfrac{1}{2}-F(\phi) \quad\text{and}\quad m_\theta = \tfrac{1}{2}+F(\phi) \tag{63c, d}$$

where

$$F(\phi) = 0{\cdot}5\tan^2(\phi/2)\cot^2(\alpha/2). \tag{64}$$

Equation (62) is the exact static collapse pressure according to an uncoupled diamond yield surface since the flow rule associated with equations (59a and 59b) requires

$$\dot{v} = 0 \quad \text{and} \quad \dot{w} = \dot{w}_0(\cos\phi - \cos\alpha)(1-\cos\alpha)^{-1} \tag{65a, b}$$

which are kinematically admissible provided $\alpha \leq \pi/2$, while equations (63a–d) are statically admissible.

If the peak value (p_0) of a dynamic pressure pulse, which is characterized by equation (4), is slightly larger than the corresponding static collapse pressure (62), then an exact solution can be obtained with the yield régime 3–5 of Fig. 2(c) which was used for the simply supported static case. The flow rules and kinematical boundary conditions remain unchanged so that the velocity yield (65a) and (65b) can again be employed for dynamic loads. Equations (59a, 59b, 65a and 65b) may be substituted into the equilibrium equations (56a) and (56c) which when integrated give

$$s\sin\phi = \gamma\ddot{w}_0(1-\cos\alpha)^{-1}\{(\sin^2\phi)/2 - \cos\alpha(1-\cos\phi)\} - (p-1)(1-\cos\phi) \tag{66}$$

and

$$m_\phi = \tfrac{1}{2} + \gamma\ddot{w}_0(1-\cos\alpha)^{-1}(1+\cos\phi)^{-1}\{2-\cos\phi-\cos^2\phi-3\cos\alpha(1-\cos\phi)\}/(6h)$$
$$-(p-1)\,(1-\cos\phi)(1+\cos\phi)^{-1}/(2h). \tag{67}$$

Now $m_\phi = 0$ at $\phi = \alpha$ so that equation (67) yields

$$\gamma\ddot{w}_0 = 3(p-p_s)/2 \tag{68}$$

where p_s is defined by equation (62). If equation (4) is substituted into (68) and the result integrated, then it is found that motion ceases at $\tau = \tau_f$ where

$$p_0(1-e^{-\tau_f}) - p_s\tau_f = 0 \tag{69}$$

while the final displacement field is $v = 0$ and

$$\gamma w_f = 3(\bar{p}\tau_f - p_s\tau_f^2/2)(\cos\phi-\cos\alpha)(1-\cos\alpha)^{-1}/2. \tag{70}$$

The total energy absorbed during the entire deformation can be expressed in the form

$$Ea = 3\pi R^2 N_0(1-\cos\alpha)p_s\tau_f(\bar{p}-p_s\tau_f/2)/(2\gamma). \tag{71}$$

It can be shown that the generalized stresses, which are obtained from the remaining equilibrium equation (56b) together with (59a, 59b, 67 and 68), may be expressed in the form

$$n_\phi = -\tfrac{1}{2} - hF, \qquad n_\theta = -\tfrac{1}{2} + hF \tag{72a, b}$$

$$m_\phi = \tfrac{1}{2} - F \quad \text{and} \quad m_\theta = -\tfrac{1}{2} + F \tag{72c, d}$$

where

$$F = \frac{(1+\cos\alpha)(1-\cos\phi)}{2(1-\cos\alpha)(1+\cos\phi)} - \frac{(p-p_s)(1-\cos\phi)(\cos\phi-\cos\alpha)}{4h(1-\cos\alpha)(1+\cos\phi)}. \tag{73}$$

This solution is kinematically admissible provided $\alpha \leq \pi/2$ and is statically admissible when $-\frac{1}{2} \leq F \leq \frac{1}{2}$, or

$$\bar{p} \leq 4h[1+\sqrt{(1+\cos\alpha)}]^2(1-\cos\alpha)^{-1} \tag{74a}$$

and

$$p_s - p_0 e^{-\gamma} \leq 4h(1-\cos\alpha)^{-1}. \tag{74b}$$

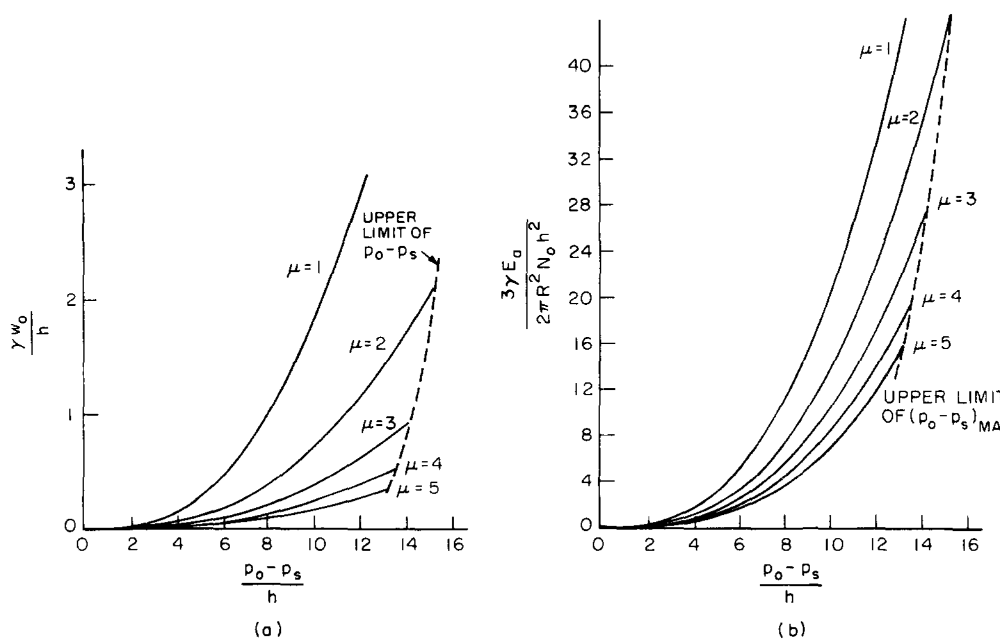
If a shell is not very shallow it has been found that the inequality (74b) is much more restrictive than (74a). Thus, the upper limit p_{0L} of a statically admissible dynamic pressure is given by the solution of the simultaneous equations (69) and (74b) with the inequality replaced by an equality.

SUMMARY OF RESULTS

The theoretical predictions of the various rigid–plastic procedures, which are outlined herein for the dynamic response of shallow and deep spherical shells loaded with an exponentially decaying pressure pulse, are presented in Figs. 3–8. These analyses are exact for the yield criteria illustrated in Fig. 2 according to the limit theorems of plasticity.

A comparison is made in Fig. 3 between the static collapse pressure and the corresponding largest peak value (p_0) of the external dynamic pressure which is permitted in the theoretical analyses of shallow spherical shells with various μ and made from materials which obey the three yield criteria shown in Figs. 2(a)–(c). The maximum permanent normal deflections of and energy absorbed by various simply supported shallow shells loaded dynamically are given in Figs. 4–7 for the same three yield conditions.

The static collapse pressure and upper limits of the peak values (p_0) of the dynamic pressure for simply supported deep spherical shells are shown in Figs. 8(a) and 8(b) for $h = 0{\cdot}005$ and $h = 0{\cdot}025$, respectively.



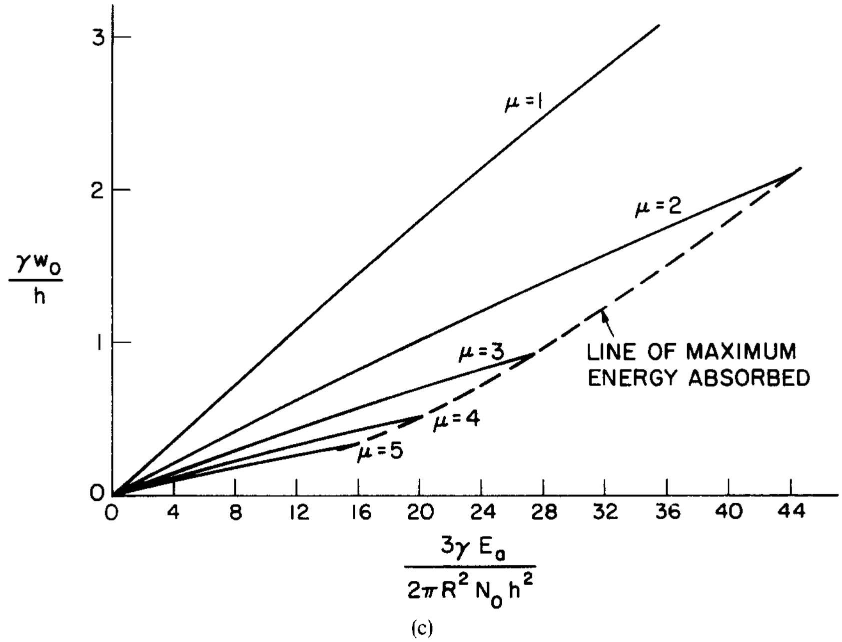


FIG. 4. Results for simply supported shallow shells which were loaded with exponentially decaying dynamic pressure pulses. Two moment limited interaction yield condition. (a) Maximum permanent normal deflections vs $p_0 - p_s$. (b) Energy absorbed during deformation vs $p_0 - p_s$. (c) Maximum permanent normal deflections vs energy absorbed during deformation.

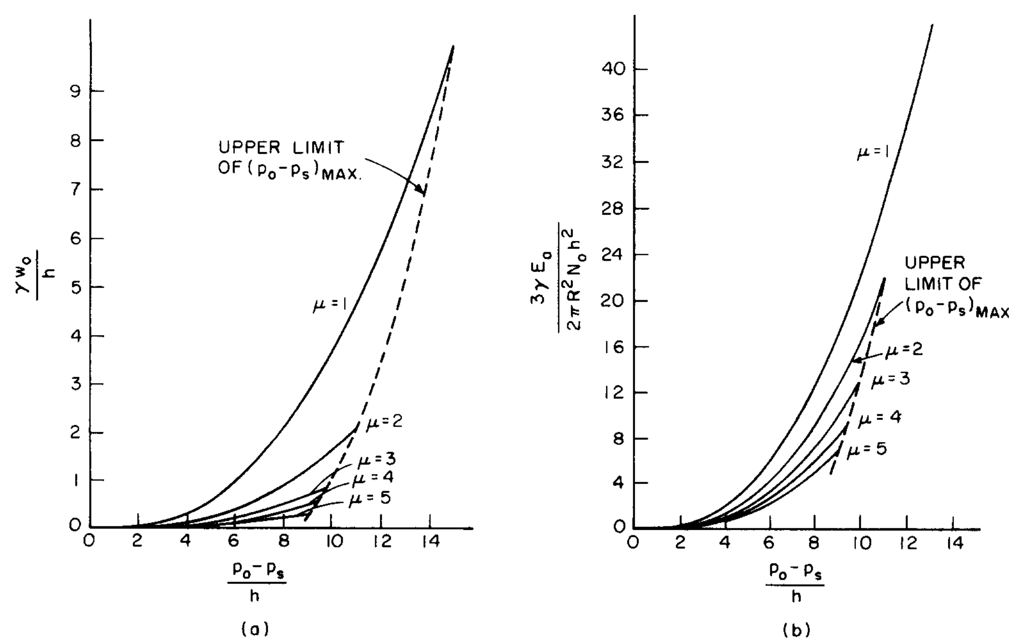


FIG. 5. Results for simply supported shallow shells which were loaded with exponentially decaying dynamic pressure pulses. Uncoupled diamond yield condition. (a) Maximum permanent normal deflections vs $p_0 - p_s$. (b) Energy absorbed during deformation vs $p_0 - p_s$.

DISCUSSION AND CONCLUSIONS

It is interesting to observe in the theoretical analysis of a shallow shell, which is based on the two moment limited interaction yield surface, that the meridional component of the velocity field is of the order (Z/R) compared to the normal component. However, in order for this analysis to be consistent, the meridional component cannot be neglected since the approximation is made that $(Z/R)^2$ is negligible compared to unity. Nevertheless, it is found that the meridional component of the velocity field is zero in an exact solution of a shallow shell which was developed with the uncoupled diamond yield surface.

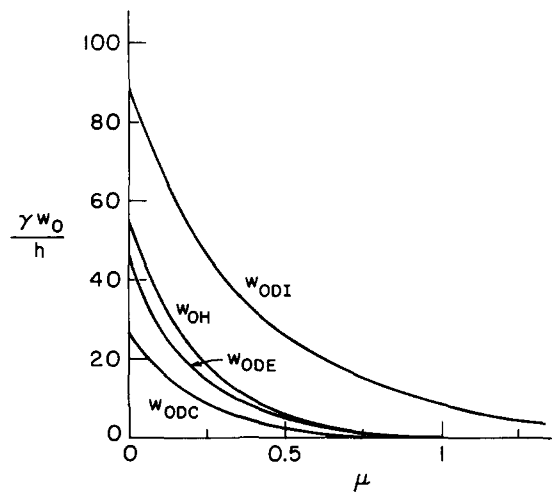


FIG. 6. Maximum permanent normal displacement vs. μ for simply supported shallow shells subjected to an exponentially decaying pressure pulse with $p_0 = 26h$.

W_{0H} predictions according to yield surface in Fig. 2(a)
W_{0DE} predictions according to yield surface in Fig. 2(c) with p_s adjusted to equal equation (6)
W_{0DI} predictions according to the yield surface in Fig. 2(c) when it inscribes Fig. 2(a)
W_{0DC} predictions according to the yield surface in Fig. 2(c) when it circumscribes Fig. 2(a)

The suggestions of Hodge and Paul [8] concerning the predictions of approximate yield surfaces in dynamic plasticity are supported by the theoretical results for shallow shells with $p_0 = 26h$, which are presented in Fig. 6. The theoretical predictions according to the two moment limited interaction yield condition may be regarded as the most accurate available. Thus, an analysis which was derived using an uncoupled diamond yield surface, with the static collapse pressure adjusted to equal equation (6), gave the best agreement with a theoretical procedure which was based on the yield condition illustrated in Fig. 2(a). An analysis which was developed with an uncoupled diamond yield condition, which was made to either inscribe or circumscribe the two moment limited interaction yield condition, respectively gave upper and lower bounds to the maximum permanent deflections which were predicted according to the yield surface in Fig. 2(a). The same remarks apply to the total energy absorbed during deformation as indicated in Fig. 7.

The validity of rigid perfectly plastic theoretical studies into the dynamic response of structures has been discussed in Refs. [9 and 10] and in other citations quoted therein. With reference to this particular article it appears from other experimental and theoretical investigations into the dynamic plastic behavior of spherical shells [11] that, unlike beams

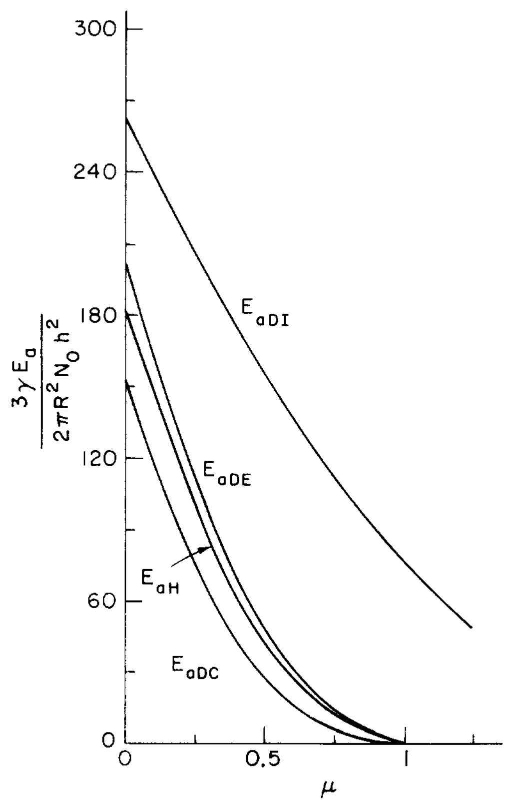
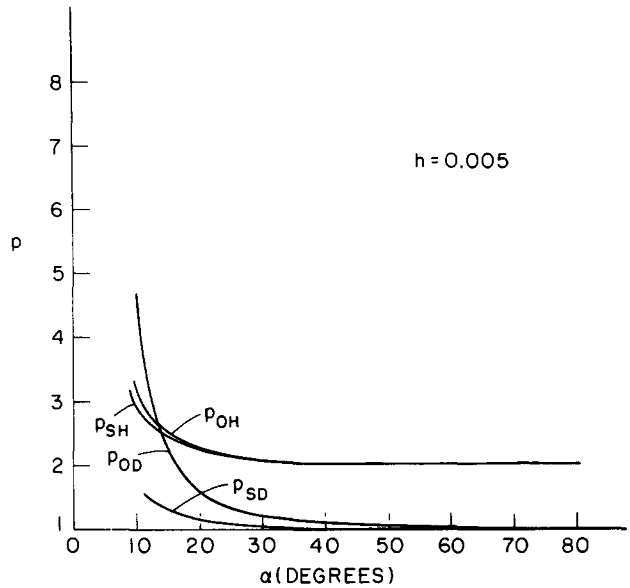


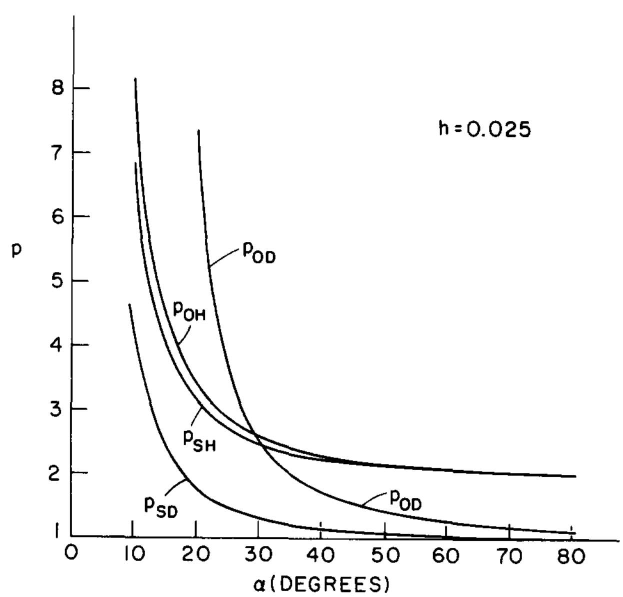
FIG. 7. Energy absorbed during deformation vs μ for simply supported shallow shells subjected to an exponentially decaying pressure pulse with $p_0 = 26h$. (E_{aH}, E_{aDE}, E_{aDI} and E_{aDC} correspond to W_{0H}, W_{0DE}, W_{0DI} and W_{0DC} defined in Fig. 6.)

and plates, the influence of finite-displacements is not important, at least for maximum normal displacements up to a few shell thicknesses.

It is evident that the uncoupled diamond yield condition simplified considerably the theoretical investigations of both shallow and deep spherical shells. However, several attempts to extend the range of validity of these analyses by retaining additional yield regimes were unsuccessful. Therefore, it appears that exact solutions, even with approximate yield surfaces, are unlikely to be found for impulsive pressure loading or for spherical shells with other boundary conditions.



(a)



(b)

FIG. 8. Static collapse pressures of simply supported deep spherical shells and upper limits of the peak values of the dynamic pressures. (a) $h = 0{\cdot}005$. (b) $h = 0{\cdot}025$

p_{sH} static collapse pressures for yield surfaces in Figs. 2(a) and 2(b)

p_{sD} static collapse pressure for yield surface in Fig. 2(c)

p_{OH} and p_{OD} upper limits of peak values of exponentially decaying dynamic pressures for yield surfaces in Figs. 2(a) and 2(c), respectively

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Абстракт—Предпринимается теоретическое исследование динамического пластического поведения свободно опертых пологих и низких сферических оболочек, подверженных действию экспоненциально затухающих импульсов давления. Предполагается, что оболочки изготовлены из жесткого, идеально пластического материала. Даются строгие решения для трех приближенных условий текучести.